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Explicit exact solutions for the generalized Zakharov equations with nonlinear terms of any order[☆]

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ABSTRACT

In this paper, an auxiliary ordinary differential equation with nonlinear terms of any order is introduced and its exact solutions are obtained. By means of the auxiliary equation and its solutions, abundant explicit exact solutions to the generalized Zakharov equations with nonlinear terms of any order are obtained in a concise manner that include new solitary wave solutions and triangular periodic wave solutions.

1. Introduction

Nonlinear evolution equations (NEEs) are widely used to describe nonlinear phenomena in various fields, such as fluid mechanics, plasma physics, solid state physics, biology and chemistry, etc. To further explain these phenomena and to apply them in the practical life, it is more important to seek their exact solutions rather than find numerical solutions, if available; exact solutions not only certify whether or not the obtained numerical solutions is better, but also are used to watch the sport rule of the wave by making the graphs of the exact solutions. In the past decades, many powerful methods have been developed such as inverse scattering method [1], homogeneous balance method [2], the mapping method [3], multilinear variable separation approach [4], Jacobi elliptic function method [5], tanh-function method [6], and so on. Unfortunately, not all these methods are universally suitable for solving all kinds of NEEs, therefore, searching for more powerful and efficient method to solve NEEs is very significant.

Recently, Sirendaoreji et al. [7–9] made use of the following auxiliary ordinary differential equation (ODE)

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 = c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi), \tag{1}$$

for solving NEEs by using auxiliary equation method. A. Elgarayhi et al. [10–12] took advantage of the following auxiliary ODE and its solutions

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 = c_2\varphi^2(\xi) + c_3\varphi^4(\xi) + c_4\varphi^6(\xi), \tag{2}$$

for constructing exact traveling wave solutions of NEEs. But Eqs. (1) and (2) can not construct explicit exact solutions to NEEs with nonlinear terms of any order in auxiliary equation method without any function transformation. In this paper,

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by introducing a unified auxiliary ODE, by means of the equation and its solutions, many new explicit exact solutions to the generalized Zakharov equations with nonlinear terms of any order are obtained in a concise manner.

2. Exact solutions of the generalized Zakharov equations with nonlinear terms of any order

At the classical level, a set of coupled nonlinear wave equations describing the interaction between high-frequency Langmuir waves and low-frequency ion-acoustic waves were firstly derived by Zakharov [13]. The Zakharov equations (ZE) may be written as

$$\begin{cases} H_{tt} - H_{xx} = (|u|^2)_{xx}, \\ \mathrm{i}u_t + u_{xx} = Hu, \end{cases} \tag{3}$$

where the complex $u(x, t)$ is envelope of the high-frequency electric field, and the real $H(x, t)$ is the plasma density measured from its equilibrium value. The ZE can be derived from a hydrodynamic description of the plasma [14,15] and is a simplified model of strong Langmuir turbulence. Thus the ZE was generalized by taking more elements into account. The generalized Zakharov equations (GZEs) are a set of coupled equations and may be written as [16]

$$\begin{cases} H_{tt} - H_{xx} = (|u|^{2\gamma})_{xx}, \\ \mathrm{i}u_t + u_{xx} = Hu + b_1 |u|^{2\gamma} u + b_2 |u|^{4\gamma} u \end{cases} \tag{4}$$

where $b_1, b_2, \gamma > 0$ are real constants, and this system is reduced to the classical Zakharov equations of plasma physics whenever $b_1 = 0, b_2 = 0, \gamma = 1$. Due to the fact that the GZEs is a realistic model in plasma, it makes sense to study the exact solutions of the GZEs.

Since $u(x, t)$ in Eq. (4) is a complex function we assume that the traveling wave solutions of Eq. (4) is the following form

$$\begin{cases} u(x, t) = a(\xi)\mathrm{e}^{\mathrm{i}[\Psi(\xi) - \omega t]} \\ H(x, t) = H(\xi), \quad \xi = x - \lambda t \end{cases} \tag{5}$$

where $a(\xi)$, $\Psi(\xi)$ and $H(\xi)$ are real functions, the constants ω and λ are to be determined.

Substituting expression (5) into Eq. (4) yields an ODE for $a(\xi)$ and $H(\xi)$.

$$(\lambda^2 - 1)H''(\xi) = (a^{2\gamma}(\xi))''. \tag{6}$$

Here and henceforth, a prime stands for the derivatives for ξ.

Integrating Eq. (6) twice and taking integration constants to zero yields

$$H(\xi) = \frac{1}{\lambda^2 - 1} a^{2\gamma}(\xi). \tag{7}$$

By means of the expression (7), yields anther ODE for $a(\xi)$ and $\Psi(\xi)$.

$$\begin{aligned} & a''(\xi) + a'(\xi)(2\mathrm{i}\Psi'(\xi) - \mathrm{i}\lambda) + a(\xi)(\omega + \lambda\Psi'(\xi) - \Psi'^2(\xi) + \mathrm{i}\Psi''(\xi)) \\ & \quad - \frac{b_1(\lambda^2 - 1) + 1}{\lambda^2 - 1} a^{2\gamma+1}(\xi) - b_2 a^{4\gamma+1}(\xi) = 0. \end{aligned} \tag{8}$$

Assume that $\Psi'(\xi) = \frac{\lambda}{2}$, Eq. (8) becomes

$$a''(\xi) + \frac{\lambda^2 + 4\omega}{4} a(\xi) - \frac{b_1(\lambda^2 - 1) + 1}{\lambda^2 - 1} a^{2\gamma+1}(\xi) - b_2 a^{4\gamma+1}(\xi) = 0. \tag{9}$$

We qualitatively analyze Eq. (9) for obtaining the kinds of the solutions of Eq. (9), letting $a'(\xi) = b(\xi)$, then Eq. (9) becomes

$$\begin{cases} b(\xi) = a'(\xi) \\ b'(\xi) = -\dfrac{\lambda^2 + 4\omega}{4} a(\xi) \left(1 - \dfrac{4}{\lambda^2 + 4\omega} \left(\dfrac{b_1(\lambda^2 - 1) + 1}{\lambda^2 - 1} a^{2\gamma}(\xi) + b_2 a^{4\gamma}(\xi)\right)\right) = -f(a(\xi)). \end{cases} \tag{10}$$

System (10) have singular points $(a(\xi), b(\xi)) = (0, 0)$ and $(a(\xi), b(\xi)) = (c, 0)$, where c satisfies $c^{2\gamma} = -\frac{1}{2b_2}(b_1 + \frac{1}{\lambda^2 - 1}) \pm \frac{1}{2b_2}\sqrt{(b_1 + \frac{1}{\lambda^2 - 1})^2 + b_2(\lambda^2 + 4\omega)}$.

For $\frac{\mathrm{d}f(a(\xi))}{\mathrm{d}a(\xi)} = \frac{\lambda^2 + 4\omega}{4} - (b_1 + \frac{1}{\lambda^2 - 1})(2\gamma + 1)a^{2\gamma}(\xi) - b_2(4\gamma + 1)a^{4\gamma}(\xi)$, in singular point $(a(\xi), b(\xi)) = (0, 0)$, we have $\frac{\mathrm{d}f(a(\xi))}{\mathrm{d}a(\xi)} = \frac{\lambda^2 + 4\omega}{4}$. Thus, when $\lambda^2 + 4\omega > 0$, $(a(\xi), b(\xi)) = (0, 0)$ is a center, system (9) have periodic Solutions. When $\lambda^2 + 4\omega < 0$, $(a(\xi), b(\xi)) = (0, 0)$ is a saddle, system (9) may have solitary solutions. We can discuss other singular points in the same way, but they are omitted. Our main goal is to seek the exact solutions of the ODE (9). We assume that $a(\xi)$ in Eq. (9) can be expressed by

$$a(\xi)=\beta\varphi^{m}(\xi) \tag{11}$$

where $\beta>0$ is constant, and $\varphi(\xi)$ satisfies the following auxiliary ODE

$$\varphi'^{2}(\xi)=c_2\varphi^{2}(\xi)+c_3\varphi^{\gamma+2}(\xi)+c_4\varphi^{2\gamma+2}(\xi), \tag{12}$$

where c_2, c_3, c_4 are constant coefficients.

Eqs. (1) and (2) are the special case of $\gamma=1$ and $\gamma=2$ in Eq. (12). By using the undetermined coefficients method [16–19], the direct method and the phase-plane analysis method [20–22], trial function method [23] and analogy technique [9,11], exact solutions for Eq. (12) are obtained. For brevity the process of solving Eq. (12) is omitted; exact solutions of Eq. (12) are as follows:

Case 1. When $c_2>0$, and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D, Eq. (12) has the following solutions:

$$\varphi_1(\xi)=\left(\frac{-c_2c_3\operatorname{sech}^2(\pm\frac{\gamma\sqrt{c_2}}{2}\xi)}{c_3^2-c_2c_4(1-\tanh(\pm\frac{\gamma\sqrt{c_2}}{2}\xi))^2}\right)^{\frac{1}{\gamma}}. \tag{13}$$

Case 2. When $c_2>0$, $c_4>0$ and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D, Eq. (12) obtains the following solutions:

$$\varphi_2(\xi)=\left(\frac{c_2\operatorname{csch}^2\frac{\gamma\sqrt{c_2}}{2}\xi}{c_3+2\sqrt{c_2c_4}\coth\frac{\gamma\sqrt{c_2}}{2}\xi}\right)^{\frac{1}{\gamma}}, \tag{14}$$

$$\varphi_3(\xi)=\left(\frac{4c_2(\cosh\gamma\sqrt{c_2}\xi+\sinh\gamma\sqrt{c_2}\xi)}{4c_2c_4-(c_3+\cosh\gamma\sqrt{c_2}\xi+\sinh\gamma\sqrt{c_2}\xi)^2}\right)^{\frac{1}{\gamma}}, \tag{15}$$

$$\varphi_4(\xi)=[(8c_2^2\operatorname{sech}\gamma\sqrt{c_2}\xi)/(c_3^2+4c_2(c_2-c_4)-4c_2c_3\operatorname{sech}\gamma\sqrt{c_2}\xi+(c_3^2-4c_2(c_2+c_4))\tanh\gamma\sqrt{c_2}\xi)]^{\frac{1}{\gamma}}, \tag{16}$$

$$\varphi_5(\xi)=\left(\frac{c_2\operatorname{csch}\frac{\gamma\sqrt{c_2}}{2}\xi}{c_3\sinh\frac{\gamma\sqrt{c_2}}{2}\xi+2\sqrt{c_2c_4}\cosh\frac{\gamma\sqrt{c_2}}{2}\xi}\right)^{\frac{1}{\gamma}}, \tag{17}$$

$$\varphi_6(\xi)=\left(\frac{c_2\operatorname{sech}\frac{\gamma\sqrt{c_2}}{2}\xi}{2\sqrt{c_2c_4}\sinh\frac{\gamma\sqrt{c_2}}{2}\xi-c_3\cosh\frac{\gamma\sqrt{c_2}}{2}\xi}\right)^{\frac{1}{\gamma}}. \tag{18}$$

Case 3. When $c_2>0$, $c_3^2-4c_2c_4>0$ and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D, Eq. (12) admits the following solutions:

$$\varphi_7(\xi)=\left(\frac{2c_2\operatorname{sech}\gamma\sqrt{c_2}\xi}{-c_3\operatorname{sech}\gamma\sqrt{c_2}\xi\pm\sqrt{c_3^2-4c_2c_4}}\right)^{\frac{1}{\gamma}}. \tag{19}$$

Case 4. When $c_2>0$, $c_3^2-4c_2c_4<0$ and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D, Eq. (12) has the following solutions:

$$\varphi_8(\xi)=\left(\frac{2c_2\operatorname{csch}\gamma\sqrt{c_2}\xi}{\pm\sqrt{4c_2c_4-c_3^2}-c_3\operatorname{csch}\gamma\sqrt{c_2}\xi}\right)^{\frac{1}{\gamma}}. \tag{20}$$

Case 5. When $c_2>0$, $c_3^2-4c_2c_4=0$ and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D, Eq. (12) obtains the following solutions:

$$\varphi_9(\xi)=\left(-\frac{c_2}{c_3}\left(1\pm\tanh\frac{\gamma\sqrt{c_2}}{2}\xi\right)\right)^{\frac{1}{\gamma}}, \tag{21}$$

$$\varphi_{10}(\xi)=\left(-\frac{c_2}{c_3}\left(1\pm\coth\frac{\gamma\sqrt{c_2}}{2}\xi\right)\right)^{\frac{1}{\gamma}}. \tag{22}$$

Case 6. When $c_2 < 0$, $c_4 > 0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq. (12) admits the following solutions:

$$\varphi_{11}(\xi) = \left(\frac{2c_2}{-c_3 \pm \sqrt{c_3^2 - 4c_2c_4} \sin \gamma \sqrt{-c_2}\xi} \right)^{\frac{1}{\gamma}}, \tag{23}$$

$$\varphi_{12}(\xi) = \left(\frac{2c_2}{-c_3 \pm \sqrt{c_3^2 - 4c_2c_4} \cos \gamma \sqrt{-c_2}\xi} \right)^{\frac{1}{\gamma}}, \tag{24}$$

$$\varphi_{13}(\xi) = \left(\frac{c_2 \sec^2 \frac{\gamma\sqrt{-c_2}}{2}\xi}{-c_3 + 2\sqrt{-c_2c_4} \tan \frac{\gamma\sqrt{-c_2}}{2}\xi} \right)^{\frac{1}{\gamma}}, \tag{25}$$

$$\varphi_{14}(\xi) = \left(\frac{c_2 \csc^2 \frac{\gamma\sqrt{-c_2}}{2}\xi}{-c_3 + 2\sqrt{-c_2c_4} \cot \frac{\gamma\sqrt{-c_2}}{2}\xi} \right)^{\frac{1}{\gamma}}, \tag{26}$$

$$\varphi_{15}(\xi) = \left(\frac{-c_2(1 + (\tan \gamma\sqrt{-c_2}\xi \pm \sec \gamma\sqrt{-c_2}\xi)^2)}{c_3 - 2\sqrt{-c_2c_4}(\tan \gamma\sqrt{-c_2}\xi \pm \sec \gamma\sqrt{-c_2}\xi)} \right)^{\frac{1}{\gamma}}, \tag{27}$$

$$\varphi_{16}(\xi) = \left(\frac{-c_2 \csc \frac{\gamma\sqrt{-c_2}}{2}\xi}{c_3 \sin \frac{\gamma\sqrt{-c_2}}{2}\xi + 2\sqrt{-c_2c_4} \cos \frac{\gamma\sqrt{-c_2}}{2}\xi} \right)^{\frac{1}{\gamma}}, \tag{28}$$

$$\varphi_{17}(\xi) = \left(\frac{c_2 \sec \frac{\gamma\sqrt{-c_2}}{2}\xi}{2\sqrt{-c_2c_4} \sin \frac{\gamma\sqrt{-c_2}}{2}\xi - c_3 \cos \frac{\gamma\sqrt{-c_2}}{2}\xi} \right)^{\frac{1}{\gamma}}. \tag{29}$$

Case 7. When $c_3 = 0$ and $(D)^{1/\gamma}$ makes sense for arbitrary negative number D, Eq. (12) has the following solutions:

$$\varphi_{18}(\xi) = \left(\pm\sqrt{\frac{c_2}{c_4}} \operatorname{csch} \gamma\sqrt{c_2}\xi \right)^{\frac{1}{\gamma}}, \quad (c_2 > 0, c_4 > 0) \tag{30}$$

$$\varphi_{19}(\xi) = \left(\pm\sqrt{-\frac{c_2}{c_4}} \operatorname{sech} \gamma\sqrt{c_2}\xi \right)^{\frac{1}{\gamma}}, \quad (c_2 > 0, c_4 < 0) \tag{31}$$

$$\varphi_{20}(\xi) = \left(\pm\sqrt{-\frac{c_2}{c_4}} \csc \gamma\sqrt{-c_2}\xi \right)^{\frac{1}{\gamma}}, \quad (c_2 < 0, c_4 > 0). \tag{32}$$

Case 8. When $c_4 = 0$ and $(D)^{1/P}$ makes sense for arbitrary negative number D, Eq. (12) obtains the following solutions:

$$\varphi_{21}(\xi) = \left(-\frac{c_2}{c_3} \operatorname{sech}^2 \frac{\gamma\sqrt{c_2}}{2}\xi \right)^{\frac{1}{\gamma}}, \quad (c_2 > 0) \tag{33}$$

$$\varphi_{22}(\xi) = \left(\frac{c_2}{c_3} \operatorname{csch}^2 \frac{\gamma\sqrt{c_2}}{2}\xi \right)^{\frac{1}{\gamma}}, \quad (c_2 > 0) \tag{34}$$

$$\varphi_{23}(\xi) = \left(-\frac{c_2}{c_3} \sec^2 \frac{\gamma\sqrt{-c_2}}{2}\xi \right)^{\frac{1}{\gamma}}, \quad (c_2 < 0). \tag{35}$$

Remark. The solutions (13)–(22), (30), (31), (33) and (34) are solitary wave solutions that include bell-profile and kink-profile solitary wave solutions, and the solutions (23)–(29), (32) and (35) are triangular periodic wave solutions. The solutions (19), (21) and (23) can be found in Refs. [16–18,23–26] and the others are new, which can not be found in literature to our knowledge.

Substituting the expression (11) and (12) into Eq. (9) and considering the homogeneous balance between $a^{4\gamma+1}(\xi)$ and $u''(\xi)$ in Eq. (9), we can obtain $(4\gamma + 1)m = 2\gamma + m$, namely $m = 1/2$, thus the expression (11) can be rewritten as

$$a(\xi) = \beta\varphi^{\frac{1}{2}}(\xi). \tag{36}$$

Substituting the expression (36) into Eq. (9) along with Eq. (12) leads to the following system of algebraic equations:

$$\lambda^2+4\omega+c_2=0,$$
$$\frac{-1-b_1(\lambda^2-1)}{\lambda^2-1}\beta^{2\gamma}+\frac{1}{4}(1+\gamma)c_3=0,$$
$$c_4(1+2\gamma)-4b_2\beta^{4\gamma}=0.$$

Solving the algebraic equations, we obtain the following results:

$$c_2=-(\lambda^2+4\omega),\qquad c_3=\frac{4}{(1+\gamma)}\left(b_1+\frac{1}{\lambda^2-1}\right)\beta^{2\gamma},\qquad c_4=\frac{4b_2}{1+2\gamma}\beta^{4\gamma}. \tag{37}$$

Substituting the expression (37) with the expressions (13)–(35) into the expressions (36), we obtain the following solitary wave solutions and triangular periodic wave solutions of Eq. (9).

$$a_1(x,t)=\left[\frac{(b_1+\frac{1}{\lambda^2-1})(\lambda^2+4\omega)\,\mathrm{sech}^2(\pm\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi)}{\frac{4}{\gamma+1}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\gamma+1)(\lambda^2+4\omega)b_2}{2\gamma+1}(1-\tanh(\pm\frac{1}{2}\gamma\sqrt{-\lambda^2-\omega}\xi))^2}\right]^{\frac{1}{2\gamma}}, \tag{38}$$

where $\lambda^2+4\omega<0$, $\xi=x-\lambda t$;

$$a_2(x,t)=\left[\frac{(-\lambda^2-4\omega)\mathrm{csch}^2(\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi)}{\frac{4}{\gamma+1}(b_1+\frac{1}{\lambda^2-1})+4\sqrt{-\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\coth(\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi)}\right]^{\frac{1}{2\gamma}}, \tag{39}$$

where $\lambda^2+4\omega<0$, $b_2>0$, $\xi=x-\lambda t$;

$$\begin{aligned}a_3(x,t)=&\left[(4\beta^{2\gamma}(\lambda^2+4\omega)(\cosh\gamma\sqrt{-\lambda^2-4\omega}\xi+\sinh\gamma\sqrt{-\lambda^2-4\omega}\xi))\Big/\left(\frac{16\beta^{4\gamma}(\lambda^2+4\omega)b_2}{2\gamma+1}\right.\right.\\&\left.\left.+\left(\frac{4}{1+\gamma}\left(b_1+\frac{1}{\lambda^2-1}\right)\beta^{2\gamma}+\cosh\gamma\sqrt{-\lambda^2-4\omega}\xi+\sinh\gamma\sqrt{-\lambda^2-4\omega}\xi\right)^2\right)\right]^{\frac{1}{2\gamma}},\end{aligned} \tag{40}$$

where $\lambda^2+4\omega<0$, $b_2>0$, $\xi=x-\lambda t$;

$$\begin{aligned}a_4(x,t)=&\left[(8\beta^{2\gamma}(\lambda^2+4\omega)^2\sec h\gamma\sqrt{-\lambda^2-4\omega}\xi)\Big/\left(\frac{16\beta^{2\gamma}(\lambda^2+4\omega)\sec h\gamma\sqrt{-\lambda^2-4\omega}\xi}{\gamma+1}\right.\right.\\&\times\left(b_1+\frac{1}{\lambda^2-1}\right)+\frac{16\beta^{4\gamma}}{(\gamma+1)^2}\left(b_1+\frac{1}{\lambda^2-1}\right)^2+4(\lambda^2+4\omega)\left(\lambda^2+4\omega+\frac{4\beta^{4\gamma}b_2}{2\gamma+1}\right)\\&\left.\left.+\left(\frac{16\beta^{4\gamma}}{(\gamma+1)^2}\left(b_1+\frac{1}{\lambda^2-1}\right)^2-4(\lambda^2+4\omega)\left(\lambda^2+4\omega-\frac{4\beta^{4\gamma}b_2}{2\gamma+1}\right)\right)\tanh\gamma\sqrt{-\lambda^2-4\omega}\xi\right)\right]^{\frac{1}{2\gamma}},\end{aligned} \tag{41}$$

where $\lambda^2+4\omega<0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_5(x,t)=\left[\frac{(-\lambda^2-4\omega)\mathrm{csch}\,\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi}{\frac{4(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\sinh\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi+4\sqrt{\frac{(-\lambda^2-4\omega)b_2}{2\gamma+1}}\cosh\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{42}$$

where $\lambda^2+4\omega<0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_6(x,t)=\left[\frac{(-\lambda^2-4\omega)\,\mathrm{sech}\,\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi}{4\sqrt{\frac{(-\lambda^2-4\omega)b_2}{2\gamma+1}}\sinh\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi-\frac{4(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\cosh\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{43}$$

where $\lambda^2+4\omega<0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_7(x,t)=\left[\frac{(\lambda^2+4\omega)\,\mathrm{sech}\,\gamma\sqrt{-\lambda^2-4\omega}\xi}{\frac{2(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\mathrm{sech}\,\gamma\sqrt{-\lambda^2-4\omega}\xi\pm2\sqrt{\frac{1}{(\gamma+1)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}}\right]^{\frac{1}{2\gamma}}, \tag{44}$$

where $\lambda^2+4\omega<0$, $\frac{1}{(1+\gamma)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{1+2\gamma}>0$, $\xi=x-\lambda t$;

$$a_8(x,t)=\left[\frac{(\lambda^2+4\omega)\operatorname{csch}\gamma\sqrt{-\lambda^2-4\omega}\xi}{\frac{2(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\operatorname{csch}\gamma\sqrt{-\lambda^2-4\omega}\xi\pm 2\sqrt{\frac{-1}{(\gamma+1)^2}(b_1+\frac{1}{\lambda^2-1})^2-\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}}\right]^{\frac{1}{2\gamma}}, \tag{45}$$

where $\lambda^2+4\omega<0$, $\frac{1}{(1+\gamma)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{1+2\gamma}<0$, $\xi=x-\lambda t$;

$$a_9(x,t)=\left[-\frac{(2\gamma+1)(b_1(\lambda^2-1)+1)}{4b_2(\gamma+1)(\lambda^2-1)}\left(1\pm\tanh\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi\right)\right]^{\frac{1}{2\gamma}}, \tag{46}$$

where $\lambda^2+4\omega<0$, $\frac{1}{(1+\gamma)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{1+2\gamma}=0$, $\xi=x-\lambda t$;

$$a_{10}(x,t)=\left[-\frac{(2\gamma+1)(b_1(\lambda^2-1)+1)}{4b_2(\gamma+1)(\lambda^2-1)}\left(1\pm\coth\frac{1}{2}\gamma\sqrt{-\lambda^2-4\omega}\xi\right)\right]^{\frac{1}{2\gamma}}, \tag{47}$$

where $\lambda^2+4\omega<0$, $\frac{1}{(1+\gamma)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{1+2\gamma}=0$, $\xi=x-\lambda t$;

$$a_{11}(x,t)=\left[\frac{\lambda^2+4\omega}{\frac{2}{\gamma+1}(b_1+\frac{1}{\lambda^2-1})\pm 2\sqrt{\frac{1}{(\gamma+1)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\sin\gamma\sqrt{\lambda^2+4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{48}$$

where $\lambda^2+4\omega>0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_{12}(x,t)=\left[\frac{\lambda^2+4\omega}{\frac{2(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\pm 2\sqrt{\frac{1}{(\gamma+1)^2}(b_1+\frac{1}{\lambda^2-1})^2+\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\cos\gamma\sqrt{\lambda^2+4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{49}$$

where $\lambda^2+4\omega>0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_{13}(x,t)=\left[\frac{(\lambda^2+4\omega)\sec^2\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}{\frac{4}{\gamma+1}(b_1+\frac{1}{\lambda^2-1})-4\sqrt{\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\tan\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{50}$$

where $\lambda^2+4\omega>0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_{14}(x,t)=\left[\frac{(\lambda^2+4\omega)\csc^2\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}{\frac{4}{\gamma+1}(b_1+\frac{1}{\lambda^2-1})-4\sqrt{\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\cot\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{51}$$

where $\lambda^2+4\omega>0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_{15}(x,t)=\left[\frac{(\lambda^2+4\omega)(1+(\tan\gamma\sqrt{\lambda^2+4\omega}\xi\pm\sec\gamma\sqrt{\lambda^2+4\omega}\xi)^2)}{\frac{4(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}-4\sqrt{\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}(\tan\gamma\sqrt{\lambda^2+4\omega}\xi\pm\sec\gamma\sqrt{\lambda^2+4\omega}\xi)}\right]^{\frac{1}{2\gamma}}, \tag{52}$$

where $\lambda^2+4\omega>0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_{16}(x,t)=\left[\frac{(\lambda^2+4\omega)\csc\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}{\frac{4(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\sin\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi+4\sqrt{\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\cos\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{53}$$

where $\lambda^2+4\omega>0$, $b_2>0$, $\xi=x-\lambda t$;

$$a_{17}(x,t)=\left[\frac{(\lambda^2+4\omega)\sec\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}{\frac{4(b_1+\frac{1}{\lambda^2-1})}{\gamma+1}\cos\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi-4\sqrt{\frac{(\lambda^2+4\omega)b_2}{2\gamma+1}}\sin\frac{1}{2}\gamma\sqrt{\lambda^2+4\omega}\xi}\right]^{\frac{1}{2\gamma}}, \tag{54}$$

where $\lambda^2 + 4\omega > 0, b_2 > 0, \xi = x - \lambda t$;

$$a_{18}(\xi) = \left(\pm\sqrt{\frac{(-\lambda^2 - 4\omega)(1+2\gamma)}{4b_2}}\operatorname{csch}\gamma\sqrt{-\lambda^2 - 4\omega}\xi\right)^{\frac{1}{2\gamma}}, \tag{55}$$

where $\lambda^2 + 4\omega < 0, b_2 > 0, \lambda = \pm\sqrt{1 - \frac{1}{b_1}}, \xi = x - \lambda t$;

$$a_{19}(\xi) = \left(\pm\sqrt{\frac{(b_1 + 4\omega b_1 - 1)(1+2\gamma)}{4b_1 b_2}}\operatorname{sech}\gamma\sqrt{-\lambda^2 - 4\omega}\xi\right)^{\frac{1}{2\gamma}}, \tag{56}$$

where $\lambda^2 + 4\omega < 0, b_2 < 0, \lambda = \pm\sqrt{1 - \frac{1}{b_1}}, \xi = x - \lambda t$;

$$a_{20}(\xi) = \left(\pm\sqrt{\frac{(\lambda^2 + 4\omega)(1+2\gamma)}{4b_2}}csc\gamma\sqrt{\lambda^2 + 4\omega}\xi\right)^{\frac{1}{2\gamma}}, \tag{57}$$

where $\lambda^2 + 4\omega > 0, b_2 > 0, \lambda = \pm\sqrt{1 - \frac{1}{b_1}}, \xi = x - \lambda t$;

$$a_{21}(\xi) = \left(\frac{(\lambda^2 + 4\omega)(1+\gamma)(\lambda^2 - 1)}{4(b_1(\lambda^2 - 1) + 1)}\operatorname{sech}^2\frac{\gamma\sqrt{-\lambda^2 - 4\omega}}{2}\xi\right)^{\frac{1}{2\gamma}}, \tag{58}$$

where $\lambda^2 + 4\omega < 0, b_2 = 0, \xi = x - \lambda t$;

$$a_{22}(\xi) = \left(-\frac{(\lambda^2 + 4\omega)(1+\gamma)(\lambda^2 - 1)}{4(b_1(\lambda^2 - 1) + 1)}\operatorname{csch}^2\frac{\gamma\sqrt{-\lambda^2 - 4\omega}}{2}\xi\right)^{\frac{1}{2\gamma}}, \tag{59}$$

where $\lambda^2 + 4\omega < 0, b_2 = 0, \xi = x - \lambda t$;

$$a_{23}(\xi) = \left(\frac{(\lambda^2 + 4\omega)(1+\gamma)(\lambda^2 - 1)}{4(b_1(\lambda^2 - 1) + 1)}\sec^2\frac{\gamma\sqrt{\lambda^2 + 4\omega}}{2}\xi\right)^{\frac{1}{2\gamma}} \tag{60}$$

where $\lambda^2 + 4\omega > 0, b_2 = 0, \xi = x - \lambda t$;

Remark. In above expressions,γ makes $(D)^{1/2\gamma}$ meaningful for arbitrary negative number D. The solutions (38)–(47), (55), (56), (58) and (59) are solitary wave solutions that include bell-profile and kink-profile solitary wave solutions, and the solutions (48)–(54), (57) and (60) are triangular periodic wave solutions.

From the expressions (5) and (7) and the solutions (38)–(60), we can write respectively the exact solutions of the GZEs, but for brevity they are omitted.

The solutions of the GZEs corresponding the expressions (44), (46) are in complete agreement with the solutions (6.7), (6.8) and (6.9) in Ref. [16], and the others. To the best of our knowledge these are unavailable in the literature.

3. Conclusions

In summary, in this paper, abundant explicit exact solutions to the GZEs with nonlinear terms of any order are obtained that include new solitary wave solutions and triangular periodic wave solutions. The key idea is taking full advantage of the solutions of auxiliary ODE, introduced in this paper. One of its advantages is to transform solving a NEE into solving an ODE whose solutions can easily be found by various methods, such as the undetermined coefficients method. The other advantage is that the auxiliary ODE (12) can be used to solve NEEs with nonlinear terms of any order, for the auxiliary Eqs. (1) and (2) are the special case of $\gamma = 1$ and $\gamma = 2$ in Eq. (12), by which it is impossible to construct explicit exact solutions for NEEs with nonlinear terms of any order in auxiliary equation method without any function transformation. The solutions (38)–(60) are the unified form of exact solutions of them. Except for the equation considered in this paper, our method also is readily applicable to a large variety of other nonlinear evolution equations including the compound KdV-type equation with nonlinear terms of any order, the generalized modified Boussinesq equation without dissipative term, the generalized one-dimensional Klein–Gordon equation, the generalized Zakharov equations, the generalized (2 + 1) dimensional Klein–Gordon equation, the Rangwala–Rao (RR) equation, the Ablowitz (A) equation and the Gerdjikov–Ivanov (GI) equation [16–19,22].

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